Borel summability of a partial propagator in a self-interacting field theory

1974 J. Phys. A: Math. Nucl. Gen. 7 356

(http://iopscience.iop.org/0301-0015/7/3/006)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 3, 1974. Printed in Great Britain. © 1974

Borel summability of a partial propagator in a self-interacting field theory

Philip B Burt

Department of Physics and Astronomy, Clemson University, Clemson, South Carolina 29631, USA

Received 19 October 1973

Abstract. The asymptotic series for a partial propagator in a self-interacting field theory, constructed from exact, particular solutions of the field equations, is shown to be Borel summable except for singularities on the light cone. The source of the divergence of the series is the occurrence of a term of the form $\langle 0|A_{\boldsymbol{k}}^{(+)2n+1}A_{\boldsymbol{q}}^{(-)2p+1}|0\rangle$, where $A_{\boldsymbol{k}}^{(\pm)}$ are annihilation and creation operators of the linear theory.

In recent years the technique of Borel summation (Whittaker and Watson 1927, Simon 1970, 1972, Graffi *et al* 1970) has been employed in the study of divergent series in perturbation theory. In particular, the Rayleigh–Schrödinger perturbation series has been shown to lead to unique energy eigenvalues for an arbitrary anharmonic oscillator through use of a generalized summability theorem developed by Graffi *et al* (1970). In this paper this theorem is used to establish the boundedness of the Borel sum for a partial propagator at points off the light cone. This propagator is constructed from exact, particular solutions of the field equations in a self-interacting field theory (Burt and Reid 1972, 1973).

The field theory considered here is described by the field equation

$$\partial_\mu\partial^\mu\phi+m^2\phi+\lambda\phi^3 = 0. \tag{1}$$

Exact particular solutions of this field equation are (Burt and Reid 1973, Reid and Burt 1973) for a system of volume V,

$$\phi_{\boldsymbol{k}}^{(\pm)} = \psi_{\boldsymbol{k}}^{(\pm)}(1+c\lambda\psi_{\boldsymbol{k}}^{(\pm)2})^{-1} \tag{2}$$

where

$$\psi_{\boldsymbol{k}}^{(\pm)} = V^{-1/2}A_{\boldsymbol{k}}^{(\pm)}\exp(\mp \mathrm{i}\check{k}\,.\,\check{x}) \tag{3}$$

$$c = -1/(8m^2) \tag{4}$$

$$\check{k}\,.\,\check{x} = k_0x_0-\boldsymbol{k}\,.\,\boldsymbol{x} \tag{5}$$

$$k_0 = \omega_k = (k^2+m^2)^{1/2} \tag{6}$$

and

$$[A_{\boldsymbol{k}}^{(+)}, A_{\boldsymbol{q}}^{(-)}] = \omega_k\delta_{\boldsymbol{k},\boldsymbol{q}}. \tag{7}$$

Using these solutions a series for the partial propagator can be found. The result is

$$P(x-y) = \int d^3k\omega_k^{-1}\omega_q^{-1}\, d^3q[\langle 0|\phi_k^{(+)}(x)\phi_q^{(-)}(y)|0\rangle\theta(x_0-y_0)$$

$$+\langle 0|\phi_k^{(+)}(y)\phi_q^{(-)}(x)|0\rangle\theta(y_0-x_0)]$$

$$= 2i\sum_{n=0}^{\infty}\left[\left(\frac{c\lambda}{V}\right)^{2n}(2n+1)!(2n+1)^{-(2n+2)}[(2n+1)^2m^2-\nabla^2]^n \times \Delta_F(x-y;(2n+1)^2m^2)\right] \tag{8}$$

where Δ_F is the Feynman propagation function (Bjorken and Drell 1965). The series has singularities on the light cone due to the presence of Δ_F. At points off the light cone it is a simple matter to show that this series is asymptotic in $(x_0-y_0, \boldsymbol{x}-\boldsymbol{y}) = (t, \boldsymbol{r})$. The argument will be illustrated for space-like intervals, where Δ_F can be written

$$\Delta_F = -(2\pi^2 r)^{-1}\frac{\partial}{\partial r}K_0[(2n+1)m\sigma]$$

$$\simeq -(2\pi^2 r)^{-1}\frac{\partial}{\partial r}\left(\frac{2(2n+1)m\sigma}{\pi}\right)^{-1/2}\exp[-(2n+1)m\sigma], \qquad \sigma \gg 0 \tag{9}$$

with

$$\sigma = (r^2-t^2)^{1/2}. \tag{10}$$

K_0 is the Bessel function of the second type with imaginary argument (Watson 1944). For arbitrary n the largest term arises from the $2n$-fold differentiation of the exponential.

This gives approximately

$$P_n \simeq (-1)^n(2n+1)!(2n+1)^{-3/2}(\tfrac{1}{2}m\pi\sigma^{-3})^{1/2}(c\lambda|t|V^{-1}\sigma^{-1}m)^{2n}\exp[-(2n+1)m\sigma]. \tag{11}$$

Thus, for fixed σ, P_n grows approximately as $(2n-1)!$, while for fixed n P_n vanishes for large σ. For timelike intervals the argument is similar, with P_n vanishing approximately as σ^{-2n} for fixed n. Consequently, the propagator is represented by an asymptotic series.

Now, it is evident that the presence of the factorial is responsible for the divergence of this series. The source of this term may be found by examining the expression for the propagator given in equation (8). The term $\langle 0|\phi_k^{(+)}(x)\phi_q^{(-)}(y)|0\rangle$ is calculated using the exact solutions of field equations given in equations (2)–(7). Suppressing the space–time dependence the vacuum expectation value

$$\langle 0|A_k^{(+)}(1+c\lambda A_k^{(+)2})^{-1}A_q^{(-)}(1+c\lambda A_q^{(-)2})^{-1}|0\rangle$$

must be calculated by expanding in series. This leads to terms of the form

$$\langle 0|A_k^{(+)2m+1}A_q^{(-)2n+1}|0\rangle = \delta_{m,n}\omega_k^{2n+1}(2n+1)! \tag{12}$$

The coefficient of this term is essentially determined by the series expansion for

$$(1+c\lambda A_k^{(+)2})^{-1}$$

—that is, by the functional form of the solution. In a linear field theory a term such as this might arise from a $(2n+1)$ particle intermediate state—but there the state $|2n+1\rangle$ would contain the normalization factor $[(2n+1)!]^{-1/2}$. In the nonlinear field theory there is no compensating term, but the (generalized) Borel summation technique can provide it.

Using the generalized Borel transform P may be written formally

$$P = \sum_{n=0}^{\infty} \frac{C_n}{(2n)!} \int_0^{\infty} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}p \exp(-p^{1/2})(4p)^{-1/2} p^n \times [(2n+1)^2 m^2 - \nabla^2]^n r^{-1} \frac{\partial}{\partial r} \exp[-(2n+1)m\sigma \cosh\phi] \tag{13}$$

where a standard generating function for K_0 has been introduced (Watson 1944) and

$$C_n = -(2n+1)!\mathrm{i}\pi^{-2}(c\lambda V^{-1})^{2n}(2n+1)^{-(2n+2)}. \tag{14}$$

While explicit evaluation of the differential operation is lengthy—the term is bounded, so we may write

$$\begin{aligned} P \simeq & \sum_{n=0}^{\infty} \frac{C_n}{(2n)!} \int_0^{\infty} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}p \exp(-p^{1/2})(4p)^{-1/2} p^n \{[(2n+1)^2 m^2 - \sigma^{-2} r^2 M (2n+1)^2 \\ & \times m^2 \cosh^2\phi]^n (2n+1) m \cosh\phi \sigma^{-1} \exp[-(2n+1) m\sigma \cosh\phi]\} \\ = & \int_0^{\infty} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}p\, (4p)^{-1/2} \exp(-p^{1/2}) m \cosh\phi \sigma^{-1} \exp(-m\sigma\cosh\phi) D \end{aligned} \tag{15}$$

where

$$D = \{[t^2 + r^2(M\cosh^2\phi - 1)]\sigma^{-2}(c\lambda m V^{-1})^2 p \exp(-2m\sigma\cosh\phi) + 1\}^{-1}, \tag{16}$$

with

$$M > 1. \tag{17}$$

Now, the expression D provides an analytic continuation of the series in p with a pole on the negative real p axis for $\sigma > 0$ and real ϕ. Consequently the integrals in p and ϕ exist. An upper bound can be established, for example, by replacing D by unity and evaluating the integrals in equation (16). Similar results can be obtained for time-like intervals.

Thus, application of the generalized Borel summation technique removes the divergence in the partial propagator in the field theory described by equations (1)–(8) at points off the light cone. The same method may be used to establish bounds on propagators for theories in which the ϕ^3 interaction is replaced by ϕ^{2q+1}, $q \neq 0, -1$, for which exact solutions are also known (Reid and Burt 1973). These results will be discussed elsewhere.

References

Bjorken J D and Drell S D 1965 *Relativistic Quantum Fields* (New York: McGraw-Hill)
Burt P B and Reid J L 1972 *J. Phys.* A: *Gen. Phys.* **5** L88
—— 1973 *J. Phys.* A: *Math., Nucl. Gen.* **6** 1388
Graffi S, Grechi V and Simon B 1970 *Phys. Lett.* **32B** 1631

Reid J L and Burt P B 1973 *J. Math. Anal. Appl.* in the press
Simon B 1970 *Phys. Rev. Lett.* **25** 1583
—— 1972 *Phys. Rev. Lett.* **28** 1145
Watson G N 1944 *A Treatise on the Theory of Bessel Functions* (Cambridge: Cambridge University Press)
Whittaker E T and Watson G N 1927 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)